

THE COARSE GEOMETRY OF THE KAKIMIZU COMPLEX

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ABSTRACT. We show that the Kakimizu complex of minimal genus Seifert surfaces for a knot in the 3-sphere is quasi-isometric to a Euclidean integer lattice \mathbb{Z}^n for some $n \geq 0$.

1. INTRODUCTION

In general, a knot $K \subset S^3$ may have multiple non-isotopic minimal genus Seifert surfaces. To understand all these possibilities, Kakimizu [5] defined a simplicial complex $\mathcal{MS}(K)$, later referred to as the *Kakimizu complex*. Each vertex σ of $\mathcal{MS}(K)$ is an isotopy class of minimal genus Seifert surfaces for K , and n -simplices are spanned by isotopy classes with pairwise disjoint Seifert surface representatives. The metric on $\mathcal{MS}(K)$ is defined by the minimal lengths of edge paths between vertices. Kakimizu [5] defined a metric on the complex using the infinite cyclic cover of K and showed that this metric is equal to the edge path metric.

Recently, $\mathcal{MS}(K)$ has been shown to be connected [11], simply connected, and contractible [8]. In fact, $\mathcal{MS}(K)$ for several classes of knots has been computed, including special arborescent knots [9] and certain composite knots [5]. Furthermore, the Kakimizu complex has been computed for all prime knots up to 10 crossings [6].

For hyperbolic knots, Pelayo [7] and Sakuma-Schackleton [10] give a bound on the diameter of the Kakimizu complex of hyperbolic knots that is quadratic in the genus of the knot, and Wilson [13] shows that $\mathcal{MS}(K)$ is finite. For satellite knots, however, $\mathcal{MS}(K)$ may be infinite, and may even be locally infinite [1].

The goal of this paper is to describe the coarse geometry of the Kakimizu complex. Recall that a *quasi-isometry* is a map $f : X \rightarrow Y$ between metric spaces X, Y such that $\frac{1}{K}d_Y(f(x), f(y)) - K \leq d_X(x, y) \leq$

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$Kd_Y(f(x), f(y)) + K$ for some constant K and every point of Y is within a K -neighborhood of the image $f(X)$. For torus and hyperbolic knots, $\mathcal{MS}(K)$ is finite and therefore quasi-isometric to a single point. For satellite knots, this simplicial structure may be more exciting.

1. Theorem. *For any knot $K \subset S^3$, the Kakimizu complex $\mathcal{MS}(K)$ is quasi-isometric to \mathbb{Z}^n for some $n \geq 0$.*

The dimension n of the abelian group can be calculated in a relatively straightforward fashion. Below, we define a subset of the complementary pieces in the JSJ decomposition for the complement of K called the core. It follows from the proof that the value of n is equal to the number of JSJ tori in the interior of the core minus the number of fibered complementary components in the core.

The outline of the paper is as follows: In Section 2, we examine how Seifert surfaces for K interact with the incompressible tori in a JSJ decomposition for the knot complement. In Section 3, we define a group action on $\mathcal{MS}(K)$ by an abelian group, generated by twisting around the tori in the JSJ decomposition, then in Section 4, we prove that this action induces a quasi-isometry from $\mathcal{MS}(K)$ to \mathbb{Z}^n .

Note that the proof here is for knots in the 3-sphere, rather than links. Przytycki-Schultens discuss ways of generalizing the Kakimizu complex to manifolds with multiple boundary components, but our proof relies on certain properties that are unique to knots, in particular the fact (proved below as a Corollary of a result proved by Ryan Budney [2]) that the Seifert fibered components of the complement of a JSJ decomposition are all small.

2. THE KAKIMIZU COMPLEX AND THE JSJ DECOMPOSITION

In [5], Kakimizu computes the Kakimizu complex for the connected sum of two non-fibered knots K_1 and K_2 with unique incompressible Seifert surfaces. In this case, $\mathcal{MS}(K_1 \# K_2)$ is isometric to $\mathbb{Z} \subset \mathbb{R}$. These Seifert surfaces come from taking the canonical Seifert surface obtained by forming the connected sum of the minimal genus Seifert surfaces for each knot and spinning it around the incompressible swallow-follow torus in the complement of the composite knot. When a knot has more than two factors, more incompressible tori would mean more ways to potentially create new Seifert surfaces by spinning.

To understand this structure, we consider the JSJ decomposition of the knot complement: let T_1, \dots, T_n be a minimal collection of pairwise disjoint, incompressible tori such that the complement of $\bigcup T_i$ consists of Seifert fibered pieces and atoroidal (hyperbolic) pieces, i.e. a JSJ decomposition. Each T_i bounds a solid torus in S^3 containing K on

one side; we will transversely orient each T_i so that the knot is on the negative side. If we consider a neighborhood $\mathcal{N}(T_i)$ of each torus and take the complement of the interior of these neighborhoods in M_K , then $M_K - \bigcup_{i=1}^n \text{int}(\mathcal{N}(T_i))$ is a collection of compact connected components called *blocks*. Because the blocks have non-empty boundary, they are Haken, so the atoroidal blocks are in fact hyperbolic. The Seifert fibered blocks are torus link complements.

2. Lemma. *Let S be a minimal genus Seifert surface for K and let $T = \bigcup_{i=1}^n T_i$ be the collection of all JSJ tori. If S is isotoped to intersect T minimally, then $|S \cap T| \leq 6g - 4$, where g is the genus of S .*

Proof. Let B_1, \dots, B_k denote the blocks of the JSJ decomposition. Each B_i is a submanifold of M_K that is either a hyperbolic link complement and hence is atoroidal, or is a torus knot or link complement and thus is Seifert fibered. Notice that S meets each B_i in a collection of orientable, essential surfaces that are properly embedded in ∂B_i .

Since S is a once punctured surface of genus g , there are at most $3g - 2$ isotopy classes of essential loops in S . Consider the components of $S \cap T$, each of which is a simple closed curve. Since both S and T are incompressible and the complement of K is irreducible, we can assume that all intersection curves are essential in both surfaces. Otherwise we could reduce the number of intersections, contradicting the minimality of $S \cap T$. In particular, all curves of $S \cap T_i$ for each i are essential in S . Therefore, there are no more than $3g - 2$ isotopy classes of curves in S for each intersection $S \cap T_i$.

In order to bound the number of components in $S \cap T$, it suffices to bound the number of parallel pairwise disjoint curves of $S \cap T_i$ for each i . Disjoint curves in $S \cap T$ that are parallel in S bound an annulus A in S , and this annulus must be incompressible and properly embedded in some block B_i . If A is boundary parallel then we could reduce $S \cap T$. Thus B_i contains an incompressible annulus so B_i can not be hyperbolic and therefore must be Seifert fibered with A isotopic to a union of fibers.

Suppose that there are three adjacent pairwise disjoint curves in $S \cap T_i$ that are parallel in S . Then the three curves correspond to two adjacent essential annuli A_1 and A_2 contained in adjacent Seifert fibered blocks. Without loss of generality, we can assume that A_1 is properly embedded in block B_1 and A_2 is properly embedded in block B_2 . Because each A_i is a union of fibers, the two fiberings of the common boundary torus T_i induced from the Seifert fiberings of B_1 and B_2 have the same slope. Therefore these two fiberings can be isotoped to agree on T_i (see [4]), contradicting the minimality of

the JSJ decomposition since T_i can be removed from the collection of JSJ tori. Therefore, there can be at most two adjacent curves of $S \cap T$ that are parallel in S in each isotopy class of curves. Hence $|S \cap T| \leq 2(3g - 2) = 6g - 4$. \square

The following is a slight generalization of the main result in [13]. The proof can be found in [13], however the statement is for manifolds with one toroidal boundary component. It is not difficult to modify the proof to also hold for manifolds with a finite number of toroidal boundary components, by a minor modification of the normal surface equations.

3. Theorem ([13]). *Let M_L be a link complement. Let $\alpha_1, \dots, \alpha_k$ be a set of preferred longitudes for the link L . If M_L contains an infinite collection of essential surfaces S_i of the same Euler characteristic such that ∂S_i is isotopic to a subcollection of the α_i and there exists an N such that $|\partial S_i| \leq N < \infty$ for each i , then M_L contains a closed incompressible torus.*

The following Corollary follows immediately from Theorem 3.

4. Corollary. *Let M_L be a link complement and $N \in \mathbf{N}$. Suppose that M_L contains no closed essential tori. Let $\alpha_1, \dots, \alpha_k$ be a set of preferred longitudes for the link L . Then M_L contains at most finitely many essential surfaces S_i of maximal Euler characteristic such that ∂S is isotopic to a subcollection of the α_i and $|\partial S_i| \leq N$ for each i .*

A Seifert fibered block of a JSJ decomposition may be toroidal, in general but the Seifert-fibered blocks of a knot complement in S^3 are much more restricted. Proposition 3.2 of [2] gives the following classification of Seifert-fibered submanifolds of S^3 .

5. Lemma. *Let $V \neq S^3$ be a Seifert-fibered sub-manifold of S^3 , then V is diffeomorphic to one of the following:*

- *A Seifert-fibered space over an n -times puncture sphere with two exceptional fibers, appearing as the complement of n regular fibers in a Seifert fibering of S^3 .*
- *A Seifert-fibered space over an n -times punctured sphere with 1 exceptional fiber, appearing as the complement of $n - 1$ regular fibers in a Seifert fibering of an embedded solid torus in S^3 .*
- *A Seifert-fibered space over an n -times punctured sphere with no exceptional fibers.*

We can use the above Lemma to show that there are only finitely many incompressible surfaces in a Seifert fibered block for a knot complement.

6. Theorem. *Let S be a minimal genus Seifert surface for a knot K and let B be a block of the JSJ decomposition for M_K . There exist finitely many essential surfaces S_1, S_2, \dots, S_m such that every component of $S \cap B$ is isotopic to one of the S_i .*

Proof. If B is a hyperbolic block, then it is atoroidal. In this case, the conclusion follows directly from Corollary 4. If B is a Seifert-fibered block, then Lemma 5 gives us three possibilities. In the first case, B is Seifert fibered over an n -times punctured sphere with two critical fibers, arising as the complement of n solid tori. Because B is a subset of a knot complement, there can be at most 1 solid torus in the complement of this block. Thus, $n = 1$ and B is Seifert-fibered over a once-punctured sphere with two critical fibers. Such Seifert-fibered spaces are known to contain no closed essential surfaces and thus are atoroidal. Applying Corollary 4, the conclusion again follows. In the second case, B is Seifert fibered over an n -punctured sphere with one critical fiber, arising as the complement of $n - 1$ solid tori. Once again, as B is a subset of a knot complement, there can be at most 1 solid torus, so $n \leq 2$. These Seifert-fibered spaces are also known to be atoroidal and thus applying Corollary 4, the conclusion follows. In the last case, B is Seifert-fibered over an n -times punctured sphere with no exceptional fibers and is thus a product. In such product spaces, there is a unique incompressible surface with boundary. \square

3. GROUP ACTIONS ON THE KAKIMIZU COMPLEX

Every automorphism of the knot complement induces an automorphism of the Kakimizu complex. For a given T_i , let U be a closed regular neighborhood homeomorphic to $I \times T_i$. We will define an automorphism of the knot complement that is the identity outside of U and spins around the JSJ torus in a given direction. Consider the universal cover of T_i , which is homeomorphic to the plane. Choose a coordinate system on this plane. For every integer vector $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, we can find a family of automorphisms ϕ_t of T_i for $t \in I$ that lift to translations of the plane in the direction of (m, n) and such that ϕ_0 and ϕ_1 are the identity on T_i . We obtain an automorphism of the knot complement as follows:

$$\Phi_i(x) = \begin{cases} (t, \phi_t(z)) & \text{if } x = (t, z) \in I \times T_i \\ x & \text{else} \end{cases}.$$

Notice that for a fixed T_i , each choice of integer vector $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ gives one of these automorphisms of the knot complement. Furthermore, composition of these automorphisms corresponds to integer vector addition in $\mathbb{Z} \times \mathbb{Z}$, which forms an abelian group of rank 2. A rank one subgroup of these automorphisms will translate along the slope of intersection $T_i \cap S$ and thus act by the identity on $\mathcal{MS}(K)$ so each torus defines a (possibly trivial) cyclic group of automorphisms of $\mathcal{MS}(K)$.

Consider G' , the group of all automorphisms of the knot complement generated by these homeomorphisms Φ_i . For $i \neq j$, the support of Φ_i is disjoint from that of Φ_j , so such homeomorphisms commute and G' is abelian. Let N be the (normal) subgroup of G' that acts trivially on the Kakimizu complex. Then $G = G'/N$, is also a finitely generated abelian group.

To prove Theorem 1, we will first prove that the result holds in the case where $\mathcal{MS}(K)$ is locally finite. (The complex is not locally finite in general [1].) To do this, we will need Theorem 25 of [3], which is stated below.

7. Theorem ([3]). *Let X be a metric space that is geodesic and proper, let G be a group and let $G \times X \rightarrow X$ be an action by isometries (say from the left). Assume that the action is proper and that the quotient $G \backslash X$ is compact.*

Then the group G is finitely generated and quasi-isometric to X . More precisely, for any $x_0 \in X$, the mapping $G \rightarrow X$ given by $g \mapsto gx_0$ is a quasi-isometry.

We will first show that $G \backslash \mathcal{MS}(K)$ is finite.

8. Lemma. *There are a finite number of minimal genus Seifert surfaces in the knot complement M_K , called fundamental surfaces, so that every Seifert surface for K is either fundamental or can be obtained by spinning a fundamental surface around some number of JSJ tori. In other words, there are a finite number of isotopy classes of minimal genus Seifert surfaces $\sigma_1, \dots, \sigma_j$ such that any other isotopy class of minimal genus Seifert surfaces $\sigma' \in \mathcal{MS}(K)$ can be written as $\sigma' = g\sigma_k$ for some $g \in G$ and some fundamental surface σ_k .*

Proof. Let S be a minimal genus Seifert surface for the knot K and assume that S meets the JSJ tori T_1, \dots, T_n transversally and minimally. Let $\mathcal{N}(T_i)$ be a neighborhood of T_i . Recall that the blocks B_k are the components of the complement of the JSJ tori.

For each k , each component of $S \cap B_k$ is isotopic to one of the finitely many possible properly embedded surfaces as given in Theorem 6. Inside each neighborhood $\mathcal{N}(T_i)$, each component of $S \cap \mathcal{N}(T_i)$ is an incompressible annulus.

Therefore, the Seifert surface S is obtained from some finite collection of the incompressible surfaces in each block by connecting these pieces with annuli across T . Up to spinning around the torus, there are finitely many ways to connect the incompressible surfaces on either side of each torus. Therefore, every Seifert surface S is in the orbit of one of finitely many isotopy classes of Seifert surfaces coming from the finite number of ways of putting together the finite components in each block. \square

Next, we will show that the action is proper.

9. Lemma. *The action of the group G on $\mathcal{MS}(K)$ is proper. That is, the stabilizer $G_v = \{g \in G \mid g(v) = v\}$ is finite for every vertex v in $\mathcal{MS}(K)$.*

Proof. Let $v \in \mathcal{MS}(K)$ be a vertex of the Kakimizu complex. To prove that G_v is finite, we will show that there is a monomorphism from G_v to a finite group.

Let O_1, O_2, \dots, O_r be the finitely many orbits of the group G . Choose a representative of each orbit $v_i \in O_i$. Let $V = \{v, v_1, \dots, v_r\}$, and let d be the diameter of V . Let B be a ball of diameter d centered at the vertex v . By construction, $V \subset B$. Notice that each automorphism of G_v preserves distance between vertices, so the ball B is fixed set-wise. Since we are assuming that $\mathcal{MS}(K)$ is locally finite, there are finitely many vertices in B . This gives a homomorphism from the stabilizer G_v to the permutation group of the (finitely many) vertices of B .

To see that this homomorphism is injective, we note that the kernel consists of all elements of G_v that fix B point-wise. Let g be such an automorphism in the kernel. Since g fixes B pointwise, then $g(v_i) = v_i$ for all i . For any $x \in \mathcal{MS}(K)$ $x = h(v_i)$ for some i , where h is some element of G . Since G is abelian, $gh = hg$, so $g(x) = g(h(v_i)) = h(g(v_i)) = h(v_i) = x$. Thus, g fixes every $x \in \mathcal{MS}(K)$. Because we quotiented out by the elements of G' that act trivially, g is the identity element in G and in G_v . So, the homomorphism from G_v to the finite permutation group is injective, and thus G_v is finite. \square

We can now combine these results to prove the following:

10. Lemma. *Let $K \subset S^3$ be a knot whose Kakimizu complex $\mathcal{MS}(K)$ is locally-finite. Then $\mathcal{MS}(K)$ is quasi-isometric to an abelian group.*

Proof. As noted above, the metric on $\mathcal{MS}(K)$ is the path metric, so the complex is properly geodesic. Since each automorphism of G takes disjoint surfaces to disjoint surfaces, it preserves distances between vertices and thus acts isometrically on $\mathcal{MS}(K)$. By Lemma 8, $G \backslash \mathcal{MS}(K)$ is finite and hence compact, and by Theorem 9, the action of G on $\mathcal{MS}(K)$ via left multiplication is proper. Thus Lemma 7 implies that $\mathcal{MS}(K)$ is quasi-isometric to G . \square

4. PROOF OF THE MAIN THEOREM

In the case when $\mathcal{MS}(K)$ is not locally finite, we will find a locally finite subcomplex that is quasi-isometric to the entire complex. A block B_i in the complement of the JSJ decomposition is a *core block* if it intersects every Seifert surface for K . We call the union of the core blocks the *core* of the JSJ decomposition of M_K , and we denote the core by C . Note that the core is nonempty since it contains the block of the JSJ decomposition that contains K . Define the *core Kakimizu complex* $\mathcal{MS}(C)$ as the subcomplex of $\mathcal{MS}(K)$ spanned by Seifert surfaces that can be isotoped entirely into C . In what follows, we will show that $\mathcal{MS}(C)$ is connected, locally finite, and is quasi-isometric to the entire complex $\mathcal{MS}(K)$. The proof of Lemma 10 applies to $\mathcal{MS}(C)$ as well, so $\mathcal{MS}(C)$ will be quasi-isometric to \mathbb{Z}^n for some $n \geq 0$; thus, $\mathcal{MS}(K)$ will also be quasi-isometric to \mathbb{Z}^n .

11. Claim. $\mathcal{MS}(C)$ is connected.

Proof. To show that the subcomplex $\mathcal{MS}(C)$ is connected, we apply the construction used by Scharlemann-Thompson [11] to find a path in the Kakimizu complex between any vertices v_1 and v_2 . Let S and S' be Seifert surfaces representing the vertices v_1 and v_2 , respectively. By taking double curve sums, Scharlemann-Thompson create a sequence of minimal genus Seifert surfaces S_i for $0 \leq i \leq k$ such that $S_i \cap S_{i+1} = \emptyset$ for $0 \leq i < k$, with S_0 isotopic to S and S_k isotopic to S' . If S and S' are both in C then so are all the double curve sums. Thus, $S_i \subset C$ for each i and the path is contained in $\mathcal{MS}(C) \subset \mathcal{MS}(K)$. \square

12. Claim. $\mathcal{MS}(C)$ is locally finite.

Proof. Let S be a minimal genus Seifert surface representing a vertex $v \in \mathcal{MS}(C)$. For any minimal genus Seifert surface $S' \subset C$ disjoint from S , there are finitely many possibilities for the intersection of S' with each core block of the JSJ decomposition by Theorem 6. The surface S' is determined by these intersections and the annuli that connect these subsurfaces inside the regular neighborhoods of the tori of T in the interior of the core. Because S intersects every component

of T in the interior of the core and S' is disjoint from S , there are finitely ways the subsurfaces can be connected together (In particular, no annulus can spin all the way around such a torus without crashing through S .) so there are finitely many minimal genus Seifert surfaces (up to isotopy) disjoint from S . \square

13. Claim. *The core of the Kakimizu complex $\mathcal{MS}(C)$ is quasi-isometric to the entire Kakimizu complex $\mathcal{MS}(K)$.*

Proof. We will show that $\mathcal{MS}(C)$ is quasi-isometric to $\mathcal{MS}(K)$ by showing that every vertex σ in $\mathcal{MS}(K)$ is within a bounded distance from some vertex σ' in $\mathcal{MS}(C)$ and the inclusion map $\mathcal{MS}(C) \hookrightarrow \mathcal{MS}(K)$ is isometric. First we note that the proof of Lemma 3 of [12] uses double curve sums to produce geodesics in $\mathcal{MS}(K)$ (as opposed to just paths). Since a double curve sum in C produces a new surface in C , this implies that the geodesics between vertices of $\mathcal{MS}(C)$ constructed in this way will be contained in $\mathcal{MS}(C)$. Thus given two vertices in the core of the Kakimizu complex $\mathcal{MS}(C)$, measuring their distance in $\mathcal{MS}(C)$ is equivalent to measuring their distance in the entire Kakimizu complex $\mathcal{MS}(K)$.

Let S be a minimal genus Seifert surface for K in the isotopy class σ . If S is contained in the core, then $\sigma \in \mathcal{MS}(C)$. If not, then, S must intersect a JSJ torus T that bounds a block B inside the core and B' outside the core. The torus T separates the Seifert surface S into a compact surface S' inside the core and finitely many annuli A_i outside the core. In fact, there are at most $3g - 2$ annuli A_i since any minimal genus Seifert surface intersects any single JSJ torus in at most $6g - 4$ circles by Lemma 2. The boundaries $C_i^+ \cup C_i^-$ of each annulus A_i can be rejoined by annuli D_i that lie inside of B . Attaching these D_i to S' yields a minimal genus Seifert surface \tilde{S} that lies completely in the core C and thus is represented by an isotopy class $\sigma' \in \mathcal{MS}(C)$.

Each time we surger the surface at a single annulus in this way, the new surface is disjoint from the previous surface. Thus σ and σ' are connected by a path in $\mathcal{MS}(K)$ of distance at most the number of annuli outside C , which is at most $3g - 2$. Thus, every $\sigma \in \mathcal{MS}(K)$ is within a bounded distance from $\mathcal{MS}(C)$ and thus the two complexes are quasi-isometric. \square

Proof of Theorem 1. By Lemma 13, $\mathcal{MS}(K)$ is quasi-isometric to the core Kakimizu complex $\mathcal{MS}(C)$. By Lemma 12, $\mathcal{MS}(C)$ is locally finite. The group G' acts on $\mathcal{MS}(C)$ in the same way that it acts on $\mathcal{MS}(K)$, so the proof of Lemma 10 can be applied to the action of G' on $\mathcal{MS}(C)$ and implies that $\mathcal{MS}(C)$ is quasi-isometric to a \mathbb{Z}^n for some

non-negative integer n . Because quasi-isometry defines an equivalence relation, this implies that $\mathcal{MS}(K)$ is quasi-isometric to \mathbb{Z}^n .

Since $\mathcal{MS}(K)$ is quasi-isometric to $\mathcal{MS}(C)$, infinite families of isotopy classes of Seifert surfaces can only be generated by spinning around tori in the interior of the core. Furthermore, if a torus bounds a fibered block to either side, one may use the fibering to undo the effect spinning. Thus, the value of the rank n in \mathbb{Z}^n is the number of JSJ tori in the interior of the core minus the number of fibered blocks in the core. \square

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